EPFL– Fall 2025
Series 6

Differential Geometry I: Curves and surfaces

G. Moschidis
17 Oct. 2025

A. Standard exercises: We will study the oriented curvature of planar curves and its geometric meaning; in what follows, if γ is a regular plane curve, we will denote κ_γ^{or} simply as κ_γ or κ. We will also begin reviewing differential calculus in many variables.

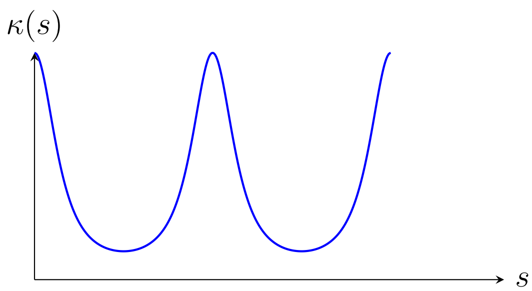
6.1 **(a)** Let γ be a plane curve whose curvature κ is a strictly monotonic function of the arc-length parameter. Can this curve be a closed C^2 curve?

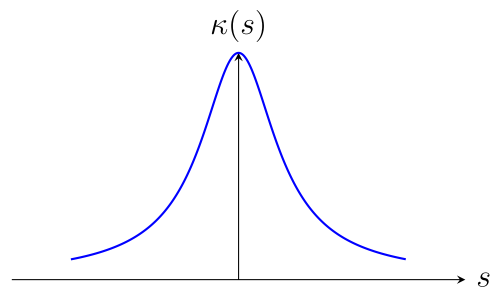
(b) Consider the following plane curves: a circle, an ellipse, and a parabola, each with its natural parametrization. For each of these curves, sketch qualitatively the graph of the function $s \mapsto \kappa(s)$ (this graph is called the *curvature diagram* of the curve).

Solution. (a) No, this cannot be a closed C^2 curve. Because, as we have seen in class, if $\gamma : [0, l] \to \mathbb{R}^2$ is a closed C^2 curve, then $\gamma(0) = \gamma(l)$, $\dot\gamma(0) = \dot\gamma(l)$ and $\ddot\gamma(0) = \ddot\gamma(l)$. In particular, since κ is defined in terms of derivatives up to order 2 of γ, we should have $\kappa(0) = \kappa(l)$. This is not possible for a strictly monotonic function (and if the $\kappa(s)$ is simply monotonic, then it has to be constant, hence the curve is a circle).

(b) Recall that the curvature diagram of a C^2 plane curve $\gamma : [a, b] \to \mathbb{R}^2$ parametrized by arc length is the graph of $\kappa(s)$.

For a circle of radius r traversed in the positive (i.e. counter clockwise) direction, $\kappa = 1/r$, so the curvature diagram is the constant function $\kappa = 1/r$. For an ellipse (again traversed in the positive direction), κ is positive and has two maxima (at the end points of the large axis) and two minima (at the end points of the small axis); see the first graph. For a parabola, viewed as the graph of the function $f(x) = x^2$ (note that this is not, however, the natural parametrization), κ is positive, has a maximum at $x = 0$ and tends to 0 as $x \to \pm\infty$; see the second graph.

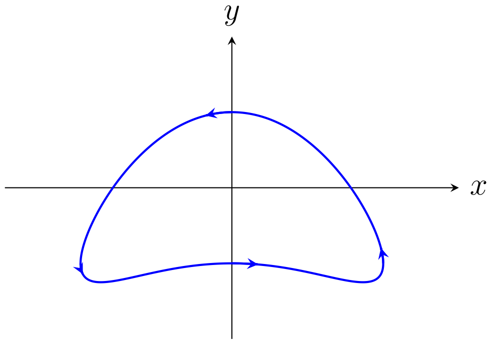


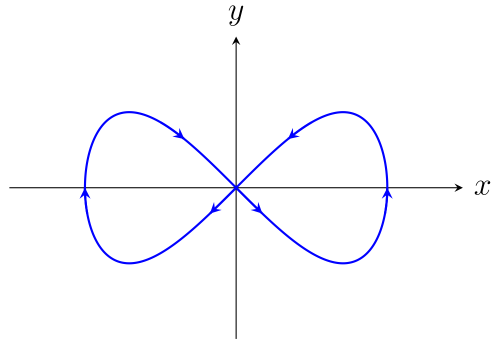


6.2 Compute the integral

$$\int_\gamma \kappa \, ds$$

for the following closed curves (parametrizaed in the direction indicated):





Solution. Recall that $\int_\gamma \kappa \, ds$ represents the total variation of the angular function φ since $d\varphi = \kappa(s)\, ds$. Therefore, for a closed curve, the above integral is simply the total number of (oriented) turns made by the tangent vector T as a point travels along the curve. For the first curve, we have that T rotates counter-clockwise exactly once, so the value is 2π. For the second curve, it is easy to see that, as we move from the point on the positive x-semiaxis to the negative x-semiaxis, the vector T returns to its original position without performing a full turn (so the total angular variation there is 0); same while moving along the other half of the curve. Thus, in the second case, the integral is equal to 0. (In general, it helps to locate inflection points and arcs where the curve is concave or convex in this process.)

6.3 Let $\gamma : I \to \mathbb{R}^2$ be a C^3 biregular planar curve. Assume that, for some $t_0 \in I$, $\gamma(t_0)$ is a vertex, i.e. $\frac{d\kappa}{dt}(t_0) = 0$ (this is property of the point $\gamma(t_0)$ is of course independent of parametrization). Show that the *evolute* β of γ is not regular at $t = t_0$. In the case when $\frac{d\kappa}{dt}(t)$ is strictly monotonic in a neighborhood of $t = t_0$, show that the unit tangent to β satisfies

$$\lim_{t \to t_0+} T_\beta(t) = - \lim_{t \to t_0-} T_\beta(t)$$

(in fact, β has a cusp point at $t = t_0$).

Solution. Recall that the evolute β of the curve γ is given by

$$\beta(t) = \gamma(t) + \frac{1}{R^2(t)} K_\gamma(t),$$

where $R(t)$ is the radius of curvature of the curve and K_γ is the curvature vector. Recall, also, that

$$R(t) = \frac{1}{|\kappa(t)|}$$

(we are using the absolute value here, since κ denotes here the oriented curvature of γ, which could take negative values; we also have that $\kappa \neq 0$, since γ was assumed to be biregular) and

$$K_\gamma(t) = \kappa(t) N_\gamma(t)$$

(where $N_\gamma(t)$ denotes the *oriented* normal of γ). Therefore,

$$\beta(t) = \gamma(t) + \frac{1}{\kappa^2(t)}\kappa(t)N_\gamma(t) = \gamma(t) + \frac{1}{\kappa(t)}N_\gamma(t).$$

Thus, by differentiating and using the Serret-Frenet formula:

$$\begin{aligned}\dot\beta(t) &= \dot\gamma + \frac{1}{\kappa(t)}\dot N_\gamma(t) - \frac{\dot\kappa(t)}{\kappa^2(t)}N_\gamma(t)\\ &= V_\gamma(t)T_\gamma(t) + \frac{1}{\kappa(t)}V_\gamma(t)\kappa(t)T_\gamma(t) - \frac{\dot\kappa(t)}{\kappa^2(t)}N_\gamma(t)\\ &= -\frac{\dot\kappa(t)}{\kappa^2(t)}N_\gamma(t)\end{aligned}$$

(this fact that should not be surprising; we have mentioned in class that the tangent line to the evolute at $\beta(t)$ is perpendicular to the original curve at $\gamma(t)$). Thus, when $\dot\kappa(t_0) = 0$, we have $\dot\beta(t_0) = 0$, i.e. $t = t_0$ is a singular point of β.

Assume, now, that $\dot\kappa(t)$ is strictly monotonic in a neighborhood of $t = t_0$. Without loss of generality, assume that it is strictly increasing (the strictly decreasing case follows in a similar way); then $\dot\kappa(t) < 0$ for $t \in (t_0 - \delta, t_0)$ and $\dot\kappa(t) > 0$ for $t \in (t_0, t_0 + \delta)$, for some $\delta > 0$. In that case, we have

$$\frac{\kappa(t)}{|\kappa(t)|} = \begin{cases} -1, & t \in (t_0 - \delta, t_0),\\ +1, & t \in (t_0, t_0 + \delta).\end{cases}$$

Therefore, since at every point where $\dot\kappa(t)$ we have

$$T_\beta(t) = \frac{\dot\beta(t)}{\|\dot\beta(t)\|} = \frac{-\frac{\dot\kappa(t)}{\kappa^2(t)}N_\gamma(t)}{\left\|\frac{\dot\kappa(t)}{\kappa^2(t)}N_\gamma(t)\right\|} = \frac{\dot\kappa(t)}{|\dot\kappa(t)|}N_\gamma(t),$$

we infer that

$$\lim_{t\to t_0+} T_\beta(t) = +N_\gamma(t) \quad \text{and} \quad \lim_{t\to t_0-} T_\beta(t) = -N_\gamma(t).$$

6.4 A bit of differential calculus:

(a) Compute the differential (in the Fréchet sense) $d\varphi_A(H)$ of the map $\varphi : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ (recall that $\mathcal{M}_n(\mathbb{R})$ is the space of $n \times n$ real matrices) defined by $\varphi(A) = A^3$, for arbitrary $A, H \in \mathcal{M}_n(\mathbb{R})$. What can be said in the special case where A and H commute?

(b) Let $\phi, \psi : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ be two differentiable maps. Prove the following version of the Leibniz rule:

$$d(\phi \cdot \psi)_A(H) = d\phi_A(H)\psi(A) + \phi(A)d\psi_A(H),$$

where $(\phi \cdot \psi)(A) = \phi(A) \cdot \psi(A)$ (matrix product).

(c) Using the previous result, show that if $\phi : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is defined by $\phi(A) = A^{-1}$, then
$$d\phi_A(H) = -A^{-1}HA^{-1}.$$

Solution. Two methods exist for computing differentials: via the definition (using the first-order Taylor expansion), or via directional derivatives. Both give the same result.

(a) By expansion:
$$\phi(A+H) = (A+H)^3 = A^3 + A^2H + AHA + HA^2 + o(\|H\|).$$

Thus
$$d\phi_A(H) = A^2H + AHA + HA^2.$$

Similarly, by finding the directional derivative:
$$\begin{aligned} d\phi_A(H) &= \frac{d}{dt}\phi(A+tH)\Big|_{t=0} = \frac{d}{dt}(A+tH)^3\Big|_{t=0} \\ &= \frac{d}{dt}\Big(A^3 + tA^2H + tAHA + tHA^2 + t^2AH^2 + t^2H^2A + t^3H^3\Big)\Big|_{t=0} \\ &= A^2H + AHA + HA^2. \end{aligned}$$

If A and H commute, the above expression simplifies to $d\phi_A(H) = 3A^2H$.

(b) Using expansions:
$$\begin{aligned} (\phi\psi)(A+H) - (\phi\psi)(A) &= \phi(A+H)\psi(A+H) - \phi(A)\psi(A) \\ &= \big(\phi(A) + d\phi_A(H) + o(\|H\|)\big)\big(\psi(A) + d\psi_A(H) + o(\|H\|)\big) - \phi(A)\psi(A) \\ &= d\phi_A(H)\psi(A) + \phi(A)d\psi_A(H) + o(\|H\|), \end{aligned}$$

so
$$d(\phi\psi)_A(H) = d\phi_A(H)\psi(A) + \phi(A)d\psi_A(H).$$

(c) Apply this to $\phi(A) = A^{-1}$ and $\psi(A) = A$. Since $(\phi\psi)(A) = I = \text{const}$, we have
$$0 = d(\phi\psi)_A(H) = d\phi_A(H)A + A^{-1}d\psi_A(H) = d\phi_A(H)A + A^{-1}H,$$

so $d\phi_A(H) = -A^{-1}HA^{-1}$.

When A and H commute, this reduces to $d\phi_A(H) = -A^{-2}H$, analogous to the scalar rule $(1/x)' = -1/x^2$.

6.5 Prove that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$$(y_1, y_2) = f(x_1, x_2) = (x_1 \cos(x_2),\, x_2 - x_1x_2)$$

is a diffeomorphism in a neighborhood of $(0,0)$.

Solution. Let us compute the Jacobian matrix:

$$Df_{(x_1,x_2)} = \begin{pmatrix} \cos x_2 & -x_1 \sin x_2 \\ -x_2 & 1 - x_1 \end{pmatrix}.$$

The Jacobian, namely the determinant of the Jacobian matrix, is

$$J_f(x_1, x_2) = (1 - x_1)\cos x_2 - x_1 x_2 \sin x_2.$$

At $(0,0)$, $J_f = 1 \neq 0$, hence $Df_{(0,0)}$ is invertible. By the inverse function theorem, we infer that there exists an open neighborhood $\mathcal{U}$ of $(0,0)$ such that f, when restricted on $\mathcal{U}$, is a diffeomorphism on its image.

6.6 **(a)** Recall the definition of a curvilinear coordinate system.

(b) Prove the following statement or find a counterexample: If $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are two curvilinear coordinate systems on an open set $U \subset \mathbb{R}^2$ and if $y_2 = x_2$, then

$$\frac{\partial}{\partial y_2} = \frac{\partial}{\partial x_2}.$$

Solution. (a) A C^k curvilinear coordinate system on $U \subset \mathbb{R}^n$ consists of n functions $y_i : U \to \mathbb{R}$ such that

$$\phi : (x_1, \dots, x_n) \mapsto (y_1, \dots, y_n)$$

is a C^k diffeomorphism from U onto an open set $V = \phi(U) \subset \mathbb{R}^n$.

(b) It is not true in general that $y_2 = x_2$ implies $\partial/\partial y_2 = \partial/\partial x_2$. Recall

$$\frac{\partial}{\partial x_2} = \frac{\partial y_1}{\partial x_2}\frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_2}\frac{\partial}{\partial y_2}.$$

Thus for equality, we must also have $\frac{\partial y_1}{\partial x_2} = 0$. A counterexample: $y_1 = ax_1 + bx_2$, $y_2 = x_2$, gives

$$\frac{\partial}{\partial x_2} = b\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}.$$

Remark: A conceptual way of deducing a counterexample as above is as follows: In general, the coordinate vector field $\frac{\partial}{\partial x^2}$ is tangent to the curves $\{x^1 = \text{const}\}$. So, as long as the level sets of x^1 and y^1 are not tangent to each other, the corresponding vector fields $\frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial y^2}$ cannot coincide.

6.7 Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be two distinct points in $\mathbb{R}^2$. Prove that the functions

$$u(x, y) = d\big((x, y), (p_1, p_2)\big), \quad v(x, y) = d\big((x, y), (q_1, q_2)\big)$$

(where $d(\cdot, \cdot)$ is the Euclidean distance in $\mathbb{R}^2$) define a C^∞ curvilinear coordinate system in each of the two half-planes bounded by the line passing through p and q. Describe the coordinate lines.

Solution. The coordinate lines are the concentric circles centered at p and at q. Let L be the line through p and q and denote by H_1 and H_2 the two half-planes determined by L. The map $(x, y) \mapsto (u, v)$ is a bijection from each half-plane H_i onto the set

$$\Omega = \{(u, v) \in \mathbb{R}^2 \mid u + v > \delta,\ u > v - \delta,\ v > u - \delta\},$$

where $\delta = d(p, q)$. Indeed, in each half-plane there is exactly one point at distance u from p and distance v from q provided $u, v > 0$.

Algebraically,

$$u(x, y) = \sqrt{(x - p_1)^2 + (y - p_2)^2}, \qquad v(x, y) = \sqrt{(x - q_1)^2 + (y - q_2)^2},$$

so u and v are C^∞ on $\mathbb{R}^2 \setminus \{p, q\}$. To show that $(x, y) \mapsto (u, v)$ is a diffeomorphism from H_i onto Ω, it suffices to check that the Jacobian never vanishes on H_i. We have

$$\begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} = \begin{pmatrix} \dfrac{x - p_1}{u(x, y)} & \dfrac{y - p_2}{u(x, y)} \\ \dfrac{x - q_1}{v(x, y)} & \dfrac{y - q_2}{v(x, y)} \end{pmatrix},$$

hence

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{u(x, y)\, v(x, y)} \det \begin{pmatrix} x - p_1 & x - q_1 \\ y - p_2 & y - q_2 \end{pmatrix}.$$

If $(x, y) \notin L$ then $u(x, y)v(x, y) > 0$ and the 2×2 determinant is nonzero because $p, q, (x, y)$ are not collinear. Thus the Jacobian does not vanish on each half-plane H_i, and (u, v) defines a C^∞ curvilinear coordinate system there.

B. Bonus exercises:

6.8 **(a)** Recall the conditions under which one can define the osculating circle of a curve $\alpha : I \to \mathbb{R}^n$ at a given point. Recall the definition of the osculating circle.

(b) How can one find the center and radius of the osculating circle at a given point of the curve? Specify the plane in which this circle lies.

(c) Prove the following result: Let $\alpha : I \to \mathbb{R}^2$ be a C^3 plane curve whose curvature is positive and strictly increasing. Then the osculating circles $\mathcal{C}(s)$ of α are nested as follows: if $s_1 < s_2$, then $\mathcal{C}(s_2)$ is contained in the disk bounded by $\mathcal{C}(s_1)$.

Hint: First show that the radius $\rho(s)$ of $\mathcal{C}(s)$ is a decreasing function of s. Then show that the distance between the centers of $\mathcal{C}(s_1)$ and $\mathcal{C}(s_2)$ is less than the difference of their radii (why does this answer the question?). To justify this last statement, it is useful to assume that α is parametrized by arc length and to compute the velocity of $s \mapsto c(s)$ (the derivative of the center of $\mathcal{C}(s)$ can be easily computed in the Frenet frame).

Solution. (a) The osculating circle is defined at any biregular point of a C^2 curve (i.e., where the velocity is nonzero and curvature is defined). The osculating circle to α at parameter t is the unique circle in the osculating plane which is tangent to α at t and whose curvature equals that of α at t. Equivalently, it is the unique circle having contact of order 2 with α at t.

(b) The radius of the osculating circle is $\rho(t) = 1/|\kappa(t)|$ and its center is

$$c(t) = \alpha(t) + \rho(t)\, N_\alpha(t),$$

where $N_\alpha(t)$ is the principal normal vector at $\alpha(t)$. The circle lies in the osculating plane at t and can be parametrized by

$$\theta \mapsto \alpha(t) + \rho(t)\big((1 - \cos\theta)N_\alpha(t) + \sin\theta\, T_\alpha(t)\big).$$

(c) Assume α is parametrized by arc length s. If $\kappa(s) > 0$ and is strictly increasing then $\rho(s) = 1/\kappa(s)$ is positive and strictly decreasing. Let $c(s) = \alpha(s) + \rho(s)N(s)$ be the center of the osculating circle. Differentiating,

$$\dot{c}(s) = \dot{\alpha}(s) + \dot{\rho}(s)N(s) + \rho(s)\dot{N}(s) = T(s) + \dot{\rho}(s)N(s) - \rho(s)\kappa(s)T(s) = \dot{\rho}(s)N(s).$$

Hence $\|\dot{c}(s)\| = |\dot{\rho}(s)| = -\dot{\rho}(s)$. For $s_1 < s_2$,

$$d\big(c(s_1), c(s_2)\big) \leqslant \int_{s_1}^{s_2} \|\dot{c}(\sigma)\|\, d\sigma = -\int_{s_1}^{s_2} \dot{\rho}(\sigma)\, d\sigma = \rho(s_1) - \rho(s_2).$$

If $x \in C(s_2)$ then

$$d\big(x, c(s_1)\big) \leqslant d\big(x, c(s_2)\big) + d\big(c(s_2), c(s_1)\big) = \rho(s_2) + \big(\rho(s_1) - \rho(s_2)\big) = \rho(s_1),$$

so x lies in the disk bounded by $C(s_1)$. This proves the nesting property.

6.9 **(a)** Let $\gamma : [0, \infty) \to \mathbb{R}^2$ be a C^3 plane curve of infinite length whose curvature is positive and strictly increasing. Prove that the trace of this curve is bounded. Can you give an explicit bound (i.e., a constant C depending maybe on the minimum curvature such that $\|\gamma(s) - \gamma(0)\| \leqslant C$ for all s)?

(b) Show by example that the monotonicity assumption on the curvature is necessary. More precisely, show that there exists a curve whose curvature satisfies $\kappa(s) \geqslant a > 0$ for all s but which is not bounded. (It is not necessary to produce an explicit formula; a drawing suffices.)

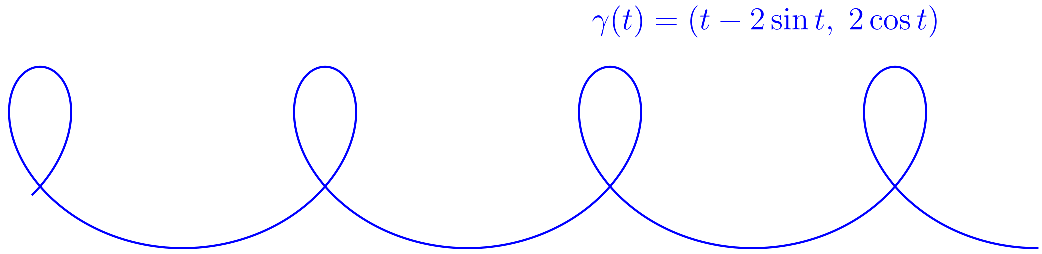
Hint for (a): Think about Exercise 6.8(c).

Solution. (a) From Exercise 6.8(c), the whole curve is contained in the osculating circle at $\gamma(0)$. Hence the image $\gamma(\mathbb{R}_+)$ is bounded and its diameter is at most $2\rho_{\gamma(0)}$.

(b) For example, the curve

$$\gamma(t) = (t - 2\sin t,\ 2\cos t)$$

has curvature bounded below by a positive constant yet goes to infinity and thus is unbounded.



6.10 Let $\gamma(s) = (x(s), y(s)) \in \mathbb{R}^2$ denote the clothoid curve parametrized by arc length (see Exercise 4.6). Do you think that the limit

$$\lim_{s\to\infty} \gamma(s) \in \mathbb{R}^2$$

exists?

(*You are asked for a geometric argument, not to compute or analyze the Fresnel integrals. Exercise 6.8 (c) is useful here.*)

Solution. The (oriented) curvature $\kappa(s)$ of the clothoid is monotone increasing and tends to $+\infty$. From Exercise 6.8(c), we obtain that, for any a, the points $\gamma(s)$ for $s \geqslant a$ are inside the osculating circle $\mathcal{C}(a)$, hence the arc $\gamma_{s\geqslant a} = \{\gamma(s) \mid s \geqslant a\}$ satisfies:

$$\operatorname{diam} \gamma_{s\geqslant a} \leqslant 2\rho(a) = \frac{2}{a}$$

and the same inequality holds for the closure $\overline{\gamma_{s\geqslant a}}$. Note that the sets $\overline{\gamma_{s\geqslant a}}$, $a \geqslant 0$, are nested, since, for $a_1 \geqslant a_2$, we have $\overline{\gamma_{s\geqslant a_1}} \subseteq \overline{\gamma_{s\geqslant a_1}}$. By the Cantor nested set principle in the complete metric space $\mathbb{R}^2$, the intersection of these nested closed sets consists of a single point, which is the limit of $\gamma(s)$ as $s \to \infty$.

Remark. Using complex analysis one can also show

$$\int_0^\infty \cos\left(\frac{s^2}{2}\right)\, ds = \int_0^\infty \sin\left(\frac{s^2}{2}\right)\, ds = \sqrt{\frac{\pi}{2}},$$

which provides another proof of the existence of $\lim_{s\to\infty} \gamma(s)$ (see e.g. the Fresnel integral).